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Glafka 2004: Spacetime Topology from the Tomographic Histories Approach I: Non-Relativistic Case

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The tomographic histories approach is presented. As an inverse problem, we recover in an operational way the effective topology of the extended configuration space of a system. This means that from a series of experiments we get a set of points corresponding to events. The difference between effective and actual topology is drawn. We deduce the topology of the extended configuration space of a non-relativistic system, using certain concepts from the consistent histories approach to Quantum Mechanics, such as the notion of a record. A few remarks about the case of a relativistic system, preparing the ground for a forthcoming paper sequel to this, are made in the end.

KEY WORDS: quantum mechanics; histories approach; topology.

1. INTRODUCTION WITH MOTIVATIONAL REMARKS

In the standard formulation of relativity theory, the spacetime topology is *a priori* fixed by the theorist to that of a continuous manifold; hence, it is not an observable entity. Only the metric structure is traditionally supposed to be dynamically variable. With the exception of Wheeler's celebrated, but largely heuristic, spacetime foam scenario (Wheeler, 1964), there is no well developed theory in which the spacetime topology can be regarded as a dynamical variable proper, with quantum traits built into the theory from the very start. However, one may try to consider idealized situations where certain topological features are represented as quantum variables that can in principle be observed and measured (Breslav *et al.*, 1999). Even in General Relativity (GR), where no variable quantity is supposed

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to be quantum—*i.e.*, subject to coherent quantum superpositions and associated uncertainty in its determinations, we need histories (*e.g.*, material particles' causal geodesic trajectories) to actually define the topology of spacetime. This is because the concept of neighborhood turns out to be something which someone (:an observer), located at some point in spacetime, deduces for regions that belong to her causal past. Similarly, the concept of distance can be established only if information (:causal signals, or actual travelling material particles) is (causally) transmitted from one point to another. All in all, the causal nexus of the world determines both its topological and metric structures.

On the other hand, an interesting feature of quantum mechanics is that we may be able to make and verify statements about topology from a single-time case, as long as we are allowed to repeat experiments (and in principle we are allowed to do that indefinitely, if only in a theoretical, idealized, 'gedanken'/theoretical fashion) in order to get the relative frequencies. In the classical (*i.e.*, non-quantum mechanical) case, one-time measurements do not give any information about global properties, such as the background topology.

Having said that, a remarkable consequence of quantum mechanics is that the wavefunction is a non-local entity, so that we may be in a position to deduce topological properties of the background, provided that we have enough repetitions of the experiment to reconstruct the relative frequencies. Thus, instead of saying that the wavefunction is a square integrable function on a topological space and use this to deduce probabilities about experimental outcomes (:events), we hereby propose to do the converse. *We start from probabilities and the continuity assumption for events, and from this information we derive the structure of the topological space in which these events are supposed to happen.* A word of caution is due here: the continuity assumption is normally taken to presuppose a topology—for how else can one talk about a *continuous* wavefunction? Well, and here is the crux of the inverse scenario: our assumption is that the wavefunction must be continuous with respect to the topology *to be deduced* from the relative frequencies of events. In other words, *the (continuity of the) wavefuction is born with the topology being deduced.*

In what follows, we do the same for the 4-dimensional case and recover 'spacetime.' We should point out that no matter that we talk about space-time, we are still in the non-relativistic regime. We just speak of space points labelled by their 'absolute,' Galilean time of occurrence. The relativistic case will be considered in a forthcoming paper (Raptis *et al.*, in press).

One could say that histories are still needed to define global properties, such as the topology; however, here we maintain instead that they are needed in order to extract the *form* of the wavefunction. Of course, *prima facie* one can counter our arguments by holding that the assumption about (complete) knowledge of the wavefunction immediately leads to EPR-type of paradoxes. Our retort is that

EPR-phenomena do not arise in our setting, while causality is rescued by the fact that we need classical communication to recover the full (complete) state, as for example in various (quantum) teleportation scenarios—see, *e.g.*, Aharonov and Vaidman (2000).

Let us outline the contents of the present paper. In the first part we introduce the consistent histories approach whereby we are given a configuration space for the system, its full Hamiltonian (including interactions), as well as the initial conditions (generally speaking, 'exosystemic' parameters of the problem traditionally supposed to be determined by an experimenter external to the experimentee—the physical system under experimental focus), and from these we calculate the probabilities for histories to occur. In our *inverse—alias*, '*tomographic*'—approach, we are given the sets of observed histories together with their relative frequencies, and from these we reconstruct (some of) the parameters of the problem, with no allusion to external/internal systemic distinctions, as befits the histories approach. Then, certain issues about topology and the character of various possible indeterminacies of the derived topology that are involved in our approach are highlighted.

The main part of the paper follows, where we present *what* we are able to recover and *how* we do that. In this paper we specifically develop the *non-relativistic* case and focus on what can be said about topology using the set of histories alone, and also what needs some further measurements in order to be 'sharply' determined. Finally, we illustrate all this by virtue of two toy-models. The first is our variant of the usual double-slit experiment, both when the particle is detected at the slit, and when it is not. The second is an example of an environment involving a 'bath of sensors.'

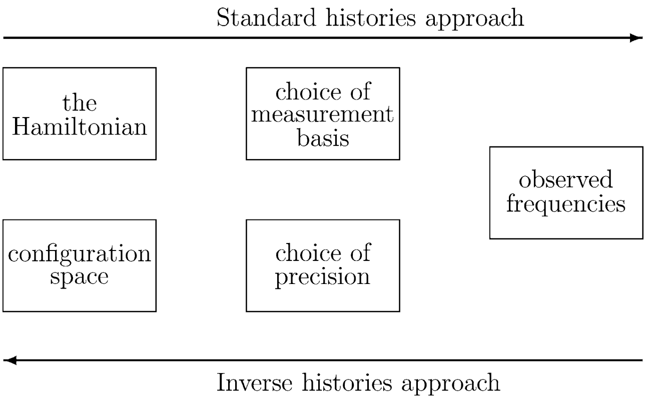
But before we delve into the paper, we feel that the new term '*tomography*' ought to be further explained; otherwise, there is no reason to have it only for the sake of fancy neologisms and lexiplacy. We believe that its use can be justified on the following semantic grounds: experiments and their records may be thought of as 'cuts' (:'$\tau o\mu \acute{\epsilon} \varsigma$ in Greek) incurred on the quantum system.[4] From (the results of) these 'observational measurement-slices' and their relative frequencies of occurrence, we 'retro-write' (:'redraw,' or 'reconstruct retrodictorily' so to speak)—as it were, 'after the fact'—the (spacetime) topology. Moreover, in Greek, the verb 'to write' (or 'to draw,' generically speaking) is $\gamma\rho\acute{\alpha}\phi\omega$. Hence '*tomo-graphy*':[5] we are re(tro)sketching spacetime topology from 'experimental cuts' exercised on the quantum system(!) All in all, this etymological dissection of the word 'tomography' accords with the title of the paper: "*spacetime topology (derived, or effectively re-sketched) from 'tomographic,' inverse histories*."

[4] Recall the Heisenberg 'schnitts' (German for 'cuts') in the standard Copenhagean quantum theory.
[5] In Greek, '$\tau o\mu o$-$\gamma\rho\alpha\phi\acute{\iota}\alpha$:='slice-wise writing/sketchching/drawing').

2. HISTORIES AND INVERSE HISTORIES APPROACH

The decoherent histories approach to quantum mechanics deals with the kind of questions that may be asked about a closed system, without the assumption of wavefunction collapse (upon measurement). It tells us, in a non-instrumentalist way, under what conditions we may meaningfully talk about statements concerning histories of our system, by using ordinary logic. This approach was mainly developed by Gell-Mann and Hartle (1990a,b,c, 1992) (Hartle, 1991a,b, 1993), and it was largely inspired by the original work of Griffiths (1984) and Omnès (1988a,b,c, 1989, 1990, 1992).

In this section, after we briefly recall useful rudiments of the standard histories scheme, we introduce its 'converse' theoretical scenario that interests us presently: *the inverse histories approach.* Pictorially, the two schemes are related as follows:



2.1. The HPO Version of the Standard Histories Approach

The formulation of the standard histories scenario that we follow presently is due to Isham *et al.* (*e.g.*, see Isham, 1994; Isham and Linden, 1994), and it is called HPO (History Projection Operator) approach. It consists of a space of histories $\mathcal{UP}$, which is the space of all possible histories of the closed system in question, and a space of decoherence functionals $\mathcal{D}$. Parenthetically, the space of histories is usually assumed to be a tensor product of copies of the standard QM Hilbert space. Two histories are called disjoint, write $\alpha \perp \beta$, if the realization of the one excludes the other. Two disjoint histories can be combined to form a third one $\gamma = \alpha \vee \beta$ (for $\alpha \perp \beta$). A complete set of histories is a set $\{\alpha_i\}$ such that $\alpha_i \perp \alpha_j \quad (\forall \alpha_i, \alpha_j, \quad i \neq j)$, and $\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_i \ldots = \mathbf{1}$

A decoherence functional is a complex valued function $d : \mathcal{UP} \times \mathcal{UP} \longrightarrow \mathbb{C}$ with the following properties:

a) Hermiticity: $d(\alpha, \beta) = d^*(\beta, \alpha)$
b) Normalization: $d(1, 1) = 1$
c) Positivity: $d(\alpha, \alpha) \geq 0$
d) Additivity: $d(\alpha, \beta \oplus \gamma) = d(\alpha, \beta) + d(\alpha, \gamma)$ for any $\beta \perp \gamma$

A complete set of histories $\{\alpha_i\}$ is said to obey the DECOHERENCE condition, *i.e.*, $d(\alpha_i, \alpha_j) = \delta_{ij} p(\alpha_i)$ while $p(\alpha_i)$ is interpreted as the probability for that history to occur *within the context of this complete set*.

The decoherence functional encodes the initial condition as well as the evolution of the system. Here we should also note that the topology of the space-time is presupposed when we group histories into complete sets, *i.e.*, in collections of partitions of unity.

In standard QM, histories correspond to time ordered strings of projections and to combination of these when they are disjoint. An important issue here is the relation between decoherence and records. Namely, it can be shown that if a set of histories decoheres, there exists a set of projection operators on the final time that are perfectly correlated with these histories and vice versa.[6] These projections are called *records*. It is this concept that figures mainly in our approach (*e.g.*, see Halliwell, 1999).

To sum things up, in the standard histories approach

- The system is given, as well as its environment. The latter is represented by prescribing initial conditions and in some cases final conditions.
- The space, its topological structure in particular, is presupposed.
- The interactions are given in terms of the decoherence functional, which encodes the dynamical information. For the complete dynamics, the full Hamiltonian must be known.

2.2. Tomographic Histories Approach

In our approach things are different, as we solve the *inverse* problem. While in standard histories one is given the Hamiltonian, initial conditions, as well as the space on which they are defined, and the aim is to predict probabilities for histories, we do the opposite thing. We make repetitions to get the frequencies for different records. Then, by making certain assumptions about these records, namely, that they are nothing but records of *events*, we recover the topological structure of the underlying configuration space. This means that from a set of events, with no other structure presupposed (:*a priori* imposed from outside the system), we end up with a causal set representing the discretized version of the *extended configuration space* of the system in question.

[6] This is the case for a *pure* initial state, and we restrict ourselves to it.

The extended configuration space that we get will be an '*effective*' one, and in a sense it accounts for certain properties of the Hamiltonian, such as interactions with other objects not controlled by the experimenter. For instance, the latter could be some kind of 'repulsive' field that prohibits the system to go somewhere (:in a region of its configuration space), which can then be recovered as a hole (:a dynamically inaccessible region) in that space.

To compare the two approaches, let us review for a moment the standard histories approach where the decoherence functional, as well as the space of histories, are given. For these, one is assumed to be given the initial conditions, the configuration space, and the Hamiltonian of the system in focus—*i.e.*, generally speaking, the parameters of the system. When we are able to perform multiple runs of the experiment and we choose a decohering set of histories, the decoherence functional yields the probability for each history to occur, which, in turn, corresponds to the history's relative frequency with respect to the set chosen.

Having the same Hamiltonian and the same initial conditions, we may consider another decohering set of histories, not necessarily compatible with the previous one, for which again the probabilities can be calculated. This is in broad terms what the usual histories approach accomplishes.

We on the other hand will be tackling the inverse problem. The essence of our approach is the following. Since we can carry out our experiment sufficiently many times, we have access to the following two things—the set of possible histories and the relative frequencies for each history to occur for every initial state. From this we recover the parameters of the experiment, namely, the effective topology of the extended configuration space.

One thing to highlight here is what corresponds to a decoherent set of histories in our inverse scenario. It is one particular partition of unity of the record space. Our freedom of choosing a particular basis in which to measure things will in general give different decoherent sets than had we chosen a different one (:different basis, different decoherent sets). Note also that since we consider histories *operationalistically*, we always deal with histories that are contained in a decoherent set, namely, the set that corresponds to the set of records that we choose to analyze.

In our setup we shall assume that *the records capture the spatio-temporal properties* (of the system in focus). This means that the histories are coarse-grained trajectories of the system, belonging to a space whose topological properties we ultimately wish to deduce. We shall then claim that the whole concept of spacetime, as a background structure, does not make sense in finer-grained situations. In this way, all the histories are single-valued on our discretized version of 'effective spacetime.' One should note here that we may still have histories that have the particle in a superposition of different position eigenstates, but only if the latter are 'finer' than the degree of our coarse-graining. With the coarse-graining we effectively identify (*i.e.*, we group into an 'equivalence class' of some sort) the points that we cannot distinguish operationally, with the resulting equivalence class of '*operationally indistinguishable points*' corresponding to a 'blown up,' 'fat point' in our discretized version of 'effective spacetime.'

2.3. Classical Versus Quantum Indeterminacy of Topology

In this subsection we would like to emphasize that there are two essentially different kinds of indeterminacy involved in derivations of the effective topology.

The first one is of a 'classical' character, that is, it comes from the lack of *our* knowledge about the systems' configuration space, as for instance when we do not have sufficiently many repetitions of the experiment. For example, the configuration space might appear to be a segment of a straight line, when in fact it is a circle. This could be due to incomplete information that we gather from an insufficiently repeated experiment, which could result to some points at the end of the segment, that would ultimately make the configuration space a circle, not to be detected. Another way that classical indeterminacy could arise would be when some records are simply not accessible[7] when, as a matter of fact, the interaction of our system with the 'record space' is supposed to capture all the spatiotemporal features or properties of the system. *In toto*, as befits the epithet 'classical,' this type of indeterminacy in effective topology determinations is an '*epistemic*' one: it reflects *our* ignorance, our partial experimental knowledge about and control over the quantum system.

The second type of indeterminacy, like the one arising in Quantum Theory, is due to a fundamental 'quantum dichotomy' of our experimental settings and determinations. For instance, the topology of coordinate and momentum space of a quantum particle may be different from each other, so that what we recover in the end depends on what we initially choose to measure: coordinates or momenta. Plainly, this reflects the fundamental quantum duality between the position and momentum observables in standard QM, which in turn is a reflection of the *ontological* (as opposed to epistemic) nature of quantum indeterminacy and uncertainty. Our setup simply limits our freedom to measure anything we want to what is produced by a decoherent set of histories, and therefore it is associated with a projection operator on our record space. We must emphasize however that we still have some freedom, since incompatible consistent sets have incompatible records in the record space, so that our choice of what basis to measure in the record space is still in force. This issue is addressed in more detail in Section 3.3.

[7] This is not the case in this paper. In our setup we assume that we have access to all records that are related to detectable events.

2.4. The Operationalistic Underpinnings of Our Scenario

Our approach is essentially *operationalistic*. The notion of *record space* is regarded as the only source of information we possess about the system we wish to explore. The effective topology then refers to the configuration space of the system in question. In our tomographic approach, we are given the sets of observed histories together with their relative frequencies, from which then we reconstruct the parameters of the problem.

We assume that some of the records may be identified with particular events, *i.e.*, spacetime 'points.' Furthermore, we claim that this is the only case we may speak of a configuration space proper. That is, if we do *not* have access to events even in principle, we *cannot* speak about their support or their topological and causal nexus, as, say, in the causal set scenario (causet). Then, relative frequencies are recovered by repetition of the whole histories involved: by restarting the system in an identical environment and letting it evolve for the same amount of time.[8] In our operationalistic (ultimately, relational-algebraic) view, the only way one can talk about some background structure such as 'spacetime,' is relative to something else. More precisely, we use our data (records) to (re)construct an 'arena' for a particular subsystem of the universe that we are interested in, and it is *only* in this sense that we may speak of 'spacetime.' Retrodictorily, *'spacetime' is where and when 'it' must have happened, if we judge by our records, and the latter are the only data we have got*. Thus, philologically speaking, '*quantum tomography is spacetime archaeology*.'

More precisely, we have a system (call it 'particle'), which is placed into an appropriate experimental environment, and we are able to

- Repeat the experiment with *the same* initial conditions. In this way we get the relative frequencies of the records.
- Vary the initial conditions of the system in question, leaving all the environment (and records) the same. For each initial condition of the system, we rerun the experiment. These first two steps give us the set of all possible histories (coarse-grained trajectories) of the particle, as well as their relative frequencies.

Another basic ingredient is the space of records. It is a space of data resulting from controlled environment tampering with the system, and it is supposed to capture its spatiotemporal properties. Records are interpreted as *distinguishable events*. That is to say,

- We can distinguish them spatiotemporally. Although we do not know the structure of the set of records that corresponds to events, we can identify each record corresponding to a spacetime point as being different from

[8] From our vantage, '*history could in principle repeat itself*' (pun intended).

the others. Thus, while we know nothing *a priori* about their causal or spatial (topological) ordering, events can be labelled so that we do not have identification problems. For instance, we may consider photons of different frequencies, each frequency mode coming from one point. In the examples to follow this will become more transparent.

- We can vary each record corresponding to a particular event independently. The variation is in some sense small—this may be effectuated by a 'small energy' variation of the record. The latter is assumed to be small enough not to affect the 'topology' of the records (*i.e.*, neighborhoods in the set of records remain the same). By 'topology' we mean a reticular structure associated with appropriate coarse-graining of a region of the extended configuration space we explore. The said variations give us the proximity relations between events.

Experiments are carried out repeatedly and multiply. We label the runs by initial conditions of the system, number of run and 'positions' of events.[9] Each run gives us a history, *i.e.*, a sequence of causally related events. Note here that for the same initial conditions of the system, the different histories group together to form decoherent sets.

To conclude, from our experiments we get the following information:

1. The set of histories of the system associated with a fixed set of initial conditions. We call this set of histories FIDUCIAL SET. Here we emphasize that these correspond to coarse-grained 'trajectories.'[10] We define the **set of all histories** to be $\mathcal{C}$, while each history that is contained in it is denoted by C_i. We therefore obtain the set $\mathcal{C}$ as well as the set $\mathcal{P}$ which is the set of all possible events, or else the set of 'spacetime' points.[11]
2. The relative frequencies of outcome of these histories depending on the initial conditions. This is a function $f_j : \mathcal{C} \to [0, 1]$ which gives the (normalized) relative frequency of histories for each particular initial condition (corresponding to jth initial state of the system).
3. The change in the relative frequencies when one event is varied. This is a function $f_j^p : \mathcal{C} \to [0, 1]$ which is the *new* relative frequencies when the event p has been varied. This will lead us to the proximity relation between the points produced by the fiducial set of histories.

It is important to note that *before* we vary the records, we already have the fiducial set of histories. It provides us the set on which the topology is imposed.

[9] By this we mean whether or not we varied one record corresponding to an event.
[10] The inverted commas are added to the word 'trajectories,' since the space on which they are defined is not presupposed.
[11] We remind here that we just speak of space points labelled by their Galilean time of occurrence.

3. NON-RELATIVISTIC CASE

We reconstruct the *effective* topology of the extended configuration space. But let us explain what we do in a bit more detail.

Effective versus 'real' topology. In our approach, we consider the effective topology which we derive from our observations. That means the following. Believing in Einstein's theory, we posit that the physical processes take place in spacetime, which is a topological space with certain *'real'* topology. However, there is no way for us to measure this 'real' topology exactly. That is why we are speaking of *effective topology*—the topology of a model of configuration space which accords with our experiments and fits their outcomes.

An important issue should be emphasized at this point. Suppose we have derived a non-trivial topology for the configuration space—say for instance that it has a defect, such as a hole. This indicates to us merely that we have non-contractible loops, nothing more. Why these loops fail to be contractible—due to the existence of a 'real hole,' or because of, say, the presence of a potential barrier—such a question is, as a matter of principle, not verifiable within our approach.

As a consequence, we may admit transitions between 3-dimensional surfaces of different number of components (with respect to the effective topology), without regarding this as being unphysical.

The record space. As noted before, we rely solely on operationalistic means to recover the effective topology. In turn, this means that we are able to control the preparation of the initial state (see Section 2.4.) and then read out the observation which is carried out by a specified device. The state space of this device we shall call RECORD SPACE. Here it should be pointed out that we assume certain things about this record space. In the case of the examples in Section 4, we specify the main features of the interaction Hamiltonian of the system with the record. More generally, we need only to assume that it captures the spatiotemporal properties of the system and therefore that it leaves records of events. Other records, outside our record space, may exist and they specify other features of the particle, such as its spin or electrical charge. If some records of the spatiotemporal properties are elsewhere then we may end up with an incomplete topology reflecting the classical indeterminacy mentioned earlier.

In closing this subsection we should also stress that since the device is anyway a quantum system, reading out the records causes some loss of information about the system in focus. Moreover, our choice of what to read out may also affect the resulting topology, which is related to the aforementioned ontological quantum indeterminacy.

3.1. Extended Configuration Space and Algebraic Considerations

We have a classical or quantum physical system, and we observe it for a period of time (t_0, t_1). If $\mathbf{M}$ is the configuration space of the system, then the

Cartesian product

$$\mathcal{M} = \mathbf{M} \times (t_0, t_1) \tag{1}$$

is the *extended configuration space*. Moreover, we also take into account a more general situation in which the topology of the configuration space **M** may change in time and the extended configuration space $\mathcal{M}$ is no longer decomposable into a product like (1)

Assume for a moment that the configuration space is at all times connected. This is not a trivial statement, as we are talking of ‘effective extended configuration space’ which in principle allows for transitions from connected to non-connected subsets in different moments of time. By considering $\mathcal{C}$, the set of all histories, we may deduce the spatial slices as the subsets S_i of points no pair of which is contained in the same history (trajectory).

$$\forall\ p, q \in S_i \Longrightarrow \nexists C_j \in \mathcal{C} \mid \quad p, q \in C_j \tag{2}$$

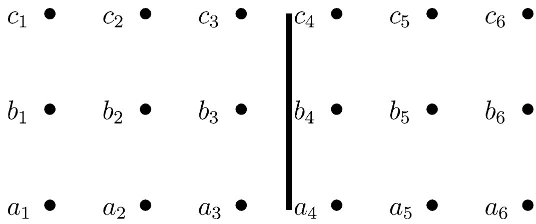
Moreover, we regard ‘maximal’ slices as being the ‘time-slices,’ *i.e.*, any extension of the spatial surface will move the subset outside the class of spatial surfaces.

$$\nexists r \in \mathcal{P} \mid r \cup S_i = S_j \tag{3}$$

Note here that the relation indicating that two points do not belong to the same history is *transitive* in the case we have only one component. It should also be noted that we cannot determine the order of the slices merely from the set of histories (*i.e.*, without varying the records), neither can we deduce any other topological feature within each of these slices. Thankfully, the latter is not the case in the relativistic situation since the upper bound in the speed of transmission of information leads to a notion of proximity in each spatial surface. This will be explored in a later publication (Raptis *et al.*, in press).

Returning to the general case, in which transitions from connected to non-connected spaces are allowed, the above procedure will produce ambiguities. Two ‘events’ could never be contained in the same history due to the fact that they are in separate connected components and *not* because they ‘occur’ at the same time. Trying then to form maximal subsets of $\mathcal{P}$ that are not contained pairwise to any history, will not lead to a unique partitioning of the set of ‘spacetime’ points . This is due to the fact that the property of two points not belonging to the same history is not transitive anymore.

An Example of Ambiguity in Partitioning: We have a non-connected space. Say we have two boxes separated by a rigid partition (*e.g.*, an infinite potential barrier). The thick line in the graph represents the partition. Apart from the obstructing partition, all histories are allowed which do not contain points of the same ‘horizontal’ line corresponding to ‘same time.’ If the particle is in one time in point a_1 at the left side of the partition, then it can never be in any of the



points on the right hand side of the partition, as *e.g.*, point b_5. This, according to the previous definition of 'time-slice,' means that a_1 is in the same slice with all the points on the right hand side of the partition no matter which instant they are measured at.

The latter would lead to contradiction, since clearly a_5 and b_5 are not in the same time-slice as there is a history joining them. If we stick to the proper definition of 'time-slice,' *i.e.*, a maximal set of points pairwise not belonging to the same history, we will end up having point a_1 in one of the following 'slices': $(a_1, a_2, a_3, a_4, a_5, a_6)$, or $(a_1, a_2, a_3, b_4, b_5, b_6)$, or, finally $(a_1, a_2, a_3, c_4, c_5, c_6)$. Any of these obey the definition of 'spatial-slice'; therefore, just from the set of histories we will end up with some ambiguity about what a spatial-slice or a 'moment of time' is.

In general, this would not be a problem since we could consider each component separately. But, in our effective set up we may have the two disconnected components becoming connected in the future. For example, the separation was made from ice and it melted (or from an unstable radioactive substance which quickly decomposed!). An example of this situation will be examined later.

We could therefore already make one non-trivial statement about the topology, just by considering the set of histories. Namely, that if there is a unique way of 'foliating' the points of the 'spacetime' into slices, the space is connected. Furthermore, we will be able to determine the number of different components of the '4-dimensional' configuration space, and on top of this, the number of components of one particular 'spatial' surface.

3.2. Extracting Connected Components

Having the set of decoherent histories, we can already extract some information about the effective topology. Let us first show how connected components are detected. In order to do this, recall that, given a connected component K of a topological space X, the relation $a\sigma b := \{a, b \in K\}$ is an equivalence relation on X.

4-Dimensional Connectedness. In our setup, we are given the relation $aHb := \{\exists C \in \mathcal{C} \mid a, b \in C\}$, which means that there exists a history containing both a and b. The relation $\simeq$ is an equivalence, *i.e.*, a symmetric, reflexive and transitive relation on X. However, the relation H is symmetric and reflexive, but not transitive. Thus, the relation σ can be obtained as the transitive closure of the relation H. In general, finding the transitive closure is an infinite operation; however, here we deal with histories containing a finite number of events, hence the transitive closure can be delimited in a finite number of steps.

A possible algorithm to find the transitive closure can be devised using Boolean matrix machinery (Zapatrin, 1994). Namely, we can define the relation H by its Boolean matrix (denote it by the same symbol H), then σ—the transitive closure of H—is obtained as a Boolean matrix power $H^{|A|}$ of H, where $|A|$ is the number of antichains. So, effectively the procedure of extracting connected components goes as follows:

- Form the Boolean matrix of the relation H 'to belong to the same history'

$$aHb := \{\exists C \in \mathcal{C} \mid a, b \in C\}$$

- Calculate its $|E|$'s power using Boolean arithmetics rather than $\oplus$ and $\otimes$:

$$\sigma = H^{|E|}$$

  Recall that the Boolean operations has the following rules: $1 + 0 = 0 + 1 = 1 + 1 = 1, 0 + 0 = 0, 1 \cdot 1 = 1, 1 \cdot 0 = 0 \cdot 1 = 0 \cdot 0 = 0$.
- The resulting matrix σ is always block-diagonal and the blocks of entries are in 1–1 correspondence with the connected components of the space E of events.

Components of a Spatial Surface. The procedure just described would account for the number of components our '4-dimensional' configuration space has. Note that, since we speak of 'effective configuration space,' we may as well have transitions, in some particular time, from a number of components to another. It would then be of interest to consider the number of components a spatial surface has.

To this end we should point out that there is some ambiguity about what a spatial surface is, thus this ambiguity will also be present in the considerations to follow.

- We let S_i be a spatial surface. $\forall\ p \in \mathcal{P}\backslash S_i$, we consider the following: $\{S_i^p \subset S_i \mid \forall\ q \in S_i^p \quad \exists \quad C_j \in \mathcal{C} \mid \quad p, q \in C_j\}$
- We will then end up with a family of subsets of S_i, call it $\mathcal{S}_{s_i}$. Note that some of these will be identical, while others may contain others. We declare them 'open.'
- From this family we generate a topology by taking arbitrary unions and finite intersections of the subsets. The resulting topology is denoted by $\mathcal{T}_{s_i}$.
- We then consider a sub-selection of the open subsets of $\mathcal{T}_{s_i}$ such that:

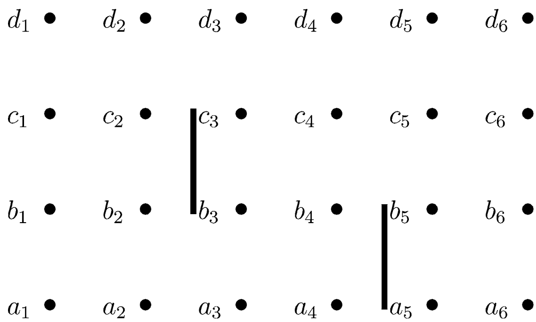
  1. It covers all S_i, *i.e.*, their union is S_i.

2. They are disjoint.
3. They are 'minimal': that is, they contain the smallest of the subsets in the family $\mathcal{T}_{s_i}$.

   This is a disjoint open covering of S_i that is also a basis for the topology $\mathcal{T}_{s_i}$.

- Finally, each of these subsets corresponds to one component of the spatial surface S_i.

To clarify things, and without wishing to repeat ourselves, we describe the above in words. We chose the surface in question. Then, for each point in space we see which part of the surface is causally connected to it. Then we pick the smallest family of subsets of the surface that covers the surface. Since the separate components do not overlap, we need to secure that this family is also disjoint. That is why we need to generate a family bigger than $\mathcal{S}_{s_i}$, namely, $\mathcal{T}_{s_i}$, while from this we are guaranteed to have a basis that consists of the relevant components, which basis would *a fortiori* be a disjoint covering.



Illustrative Example of Recovering the Components of a Spatial Surface: Here the space is connected when seen 4-dimensionally. The partitions that exist forbid for example a history containing a_5 and b_3 (the lower one), or a_5 and c_1 (the higher partition). Note that even without the 'd-column,' the space is connected when viewed '4-dimensionally,' as the 'transitive closure' of any point is the set itself.

Now we follow the steps described above. We pick the spatial slice that corresponds to the b-horizontal line $(b_1, b_2 \ldots, b_6)$, and we are looking for its components.

First we consider the set $\mathcal{S}_{s_i}$, which in this case is the set containing the following subsets: $\{(b_1, b_2), (b_1, b_2, b_3, b_4), (b_3, b_4, b_5, b_6), (b_5, b_6), (b_1, b_2, b_3, b_4, b_5, b_6)\}$. Note that the subset (b_3, b_4) does not belong to $\mathcal{S}_{s_i}$. The result we would like to have is that there are three components, namely, $\{(b_1, b_2), (b_3, b_4), (b_5, b_6)\}$. To obtain this, we have to follow Section 3.2. and extend $\mathcal{S}_{s_i}$ to $\mathcal{T}_{s_i}$, which is the topology induced by $\mathcal{S}_{s_i}$ if we consider unions and intersections. In the latter, the subset (b_3, b_4) is also included as it is the intersection of (b_1, b_2, b_3, b_4) and (b_3, b_4, b_5, b_6).

We then need to pick a sub-selection of the elements of $\mathcal{T}_{s_i}$ that is disjoint and covers the surface (*i.e.*, the horizontal b). There are two possible choices: either $\{(b_1, b_2), (b_3, b_4), (b_5, b_6)\}$, or $\{(b_1, b_2, b_3, b_4, b_5, b_6)\}$. The second is not 'minimal,' *i.e.*, it does not contain the smallest sets and therefore it is not a basis for the topology $\mathcal{T}_{s_i}$. Finally, we are left with $\{(b_1, b_2), (b_3, b_4), (b_5, b_6)\}$, which is the desired result.

A final note just to mention that the above discussion is liable to ambiguities that come from the fact that there is not a unique definition of spatial surface. Instead of the b-horizontal as a surface, we could have taken as spatial surface for example the subset $\{(c_1, c_2, b_3, b_4, b_5, b_6)\}$, and we would end up with similar results.

3.3. Reconstruction of Topology—Statistical Approach

As mentioned earlier, different decohering sets of histories may lead us to different effective topologies. It follows that the effective topology is a result of our measurements. We could claim that our system is in a superposition of different effective 'spacetimes'[12] and our choice of measurement causes a 'reduction' to one particular (or to a particular subspace of all the possible) 'spacetime.' In the description above we carry out a measurement in record space on the 'basis' that is related with spacetime points, *i.e.*, events. If this assumption is not satisfied, the actual choice of our measurements would affect the resulting topology. It should be pointed out here that this is the generic case, since we cannot have full knowledge about whether or not our records capture only configuration space properties and not, possibly incompatible, momentum space as well. On the other hand, if our measurements are sufficiently coarse, we could have compatible 'position' and 'momentum' measurements.

Now we are in a position to address how to recover topology assuming that we can vary slightly one event independently from the others, and repeat the runs of the experiment. The result of such variations will be certain changes of the relative frequencies, that is why we call this process **statistical** reconstruction of topology. This procedure fixes the ambiguities about the 'time-slices' that existed due to the non-connected spatial surfaces, as well as the order of these slices.

- We have the relative frequencies, $f_j(C_i)$ of each history C_i with initial condition'j.'
- We **vary slightly one event** say event $p \in \mathcal{P}$ and repeat the procedure to get the new relative frequencies of histories $f_j^p(C_i)$. It is important to

[12] Here we still assume that we are in the non-relativistic regime.

note that, provided the variation is small, the set of histories is the same and only their relative frequencies change. Therefore, all the considerations that were already made from the mere set of histories still apply galore.

By observing which histories have changed their frequencies compared to the undisturbed event case, we can deduce a few things—for starters, some notion of closeness (or proximity). The histories whose frequencies are significantly affected by the perturbation are in some sense 'close.'

- We consider *each initial condition separately*.[13] For each initial condition we see the **probabilities of which histories alter significantly**.
- After we vary the 'event,' we repeat the experiment exactly with same initial condition as before and then, by considering the change in relative frequency of events (not of histories), we can **deduce which events are neighbors** (call them j-neighbors, whereby the label 'j' stands for the initial condition we consider). We repeat this for all possible initial conditions. We then consider the union of all these j-neighborhoods to get the total neighborhood of the point we varied.

In other words we take a small positive number $\epsilon \ll 1$. We define another function, the difference function, as follows:

$$\delta_j^p : \mathcal{C} \to [0, 1] :| f_j(C_i) - f_j^p(C_i) | \tag{4}$$

We then consider all the points belonging to the histories $C_i \in \mathcal{C}$ that $\delta_j^p(C_i) > \epsilon$. We name them j-neighbors of p. So we have:

$$q \in N_j^p \Longrightarrow \exists \quad q \in C_i, C_i \in \mathcal{C} \mid \delta_j^p(C_i) > \epsilon \tag{5}$$

We then consider different initial conditions 'j' and we group all the neighbors together to form the neighbors of 'p' , N^p.

$$q \in N^p \Longrightarrow \exists \quad j \mid q \in N_j^p \tag{6}$$

- We already know which of these neighbors are (definitely) not in the same time-slice (:those that both belong to at least one history), and we can coin them **'temporal neighbors.'** Events being in different path-connected components will never affect each other. Note here that the neighbors that will be affected, and are definitely not in the same time, are only to the future of the event in question. Thus, properly speaking, we should talk

[13] This to avoid problems related with the following. Assume that we vary a point a in a way that it has the same distance with one neighboring point b. Then the overall probability of the b due to symmetry will be invariant, but depending on which is the initial condition of the system the probabilities of some histories will increase while other will decrease with a net probability unchanged. In this way we would fail to recognize b as a neighbor of a.

about **'future temporal neighbors.'** With these in hand, we may get the order of the histories.[14]

- Then we mark the events that are neighbors, but not *temporal* neighbors, as **'spatial-neighbors,'** and use them to define proximity in the 'time-slice' in focus.

  So we define *spatial neighborhood* of 'p' to be:

$$SN^p \mid q \in [N^p \setminus \cup_i C_i] \quad , \quad p \in C_i \quad \forall\, i \tag{7}$$

- We repeat this procedure varying slightly one by one all the 'events.'
- From the proximity we deduce the topology of each time slice in the usual way—*e.g.*, as it is done in metric spaces.
- We will have obtained the topology of each spatial components. We can then choose an arbitrary partitioning of these slices to get the total 4-dimensional case. We then check that we do not have contradiction.This contradiction could be due to, for example, some event being affected by a change in an event to its future rather than to its past (:'advanced' and 'retarded' contradiction, respectively).If a contradiction arises, we pick another 'partitioning,' so on and so forth, until the correct one is obtained.
- By patching all the slices together, we **recover the topology** of our 'space-time,' or more precisely, of its reticular substitute .

Alternatively, we may consider the closest neighbors to define a cover of each time-slice, and then find the finitary substitute of the underlying continuous topology. To find only the closest neighbors, we need to 'tune' the parameter 'ϵ' to be sufficiently big so that it gives only the number of closest neighbors we want (4 to get a 3-dimensional space in the triangulations scheme). Along these lines, we first get a *prebase* from which the topology is unambiguously reconstructed. We should note here that the above construction makes 'heavy' use of the relative frequencies, not only of the set of histories. It effectively uses the former to define neighborhoods.

4. TOY MODELS

4.1. Double-Slit Experiment

We consider a discrete version of the registration screen. This means that our data will be a discrete distribution of registered events, each 'column' being discretely labelled. We consider the case that we do not detect which slit the particle passes through, as well as the case that we do. In both cases we have always the same initial conditions—a particle is emitted far away from a barrier

[14] Note that since we get a direction from the fact that only the 'future' neighbors are affected, helps us recover the order with no doubt about the overall direction.

bearing two holes. Note also that the particle in question is assumed not to be a photon, so that it can be detected on the slit without being absorbed.

Case I: Not Detected on the Slit. The particle passes through the slit. Then it is absorbed by a film so that we can identify different events by measuring the position of the excited grain on the film. Note that we need only distinguish the different events and not their actual position. So, somebody could have cut the film and glued it back with different order (but the same for all the repetitions). The correspondence between this gedanken-experimental scenario and our theoretical scenario above is the following:

- single experiment—emitting one particle and registering it
- the record space—the real line (position-*loci* of registered events)
- a particular history—an event

To recover the configuration space, we (a) assume continuity of the distribution, and (b) move slightly one point of registration on the film. By observing the probabilities of the events that are significantly altered, we define the neighborhood of this 'point' (:proximity neighborhood).

We recover several segments of a line representing the configuration space. The fact that it is not the whole real line that is being recovered, is due to the fact that there exist dark fringes, *i.e.*, regions where the particle is never detected.

Case II: Detected on the Slit. The particle passes through the slit, and a photon, whose frequency depends on which slit the particle passes through, is emitted. This happens because we have an oscillator of different frequency on each slit, and when a particle passes, the oscillator increases its energy level. Then, upon relaxing back to its ground state, it emits a photon. Then the particle is absorbed by the discrete screen. The correspondence between this gedanken-experimental scenario and our theoretical scenario above is the following:

- single experiment—emitting one particle, and subsequently registering it as well as the photon carrying information about which slit the particle passed.
- the record space—the real line (position-*loci* of registered events) and the detector of the photon (or the oscillators). Note here that we can distinguish all events from each other, but not know anything else about their topological structure.
- a particular history—a photon with frequency depending on which slit the particle passes through, followed by a position on the discrete screen line.

To recover the configuration space, we (a) assume continuity of the distribution, and (b) we move slightly one point of registration on the film, or one of the oscillators. As usual, we recover neighborhoods by small variations of the established 'records.'

What is Eventually Recovered. Two points separated from one another, and at a later time[15] a segment of straight line (actually a syncopated version of it). Note that the line does not decay into disjoint segments, as we have no interference and therefore there are no 'dark fringes.'

4.2. Bath of Sensors

Here we consider a thought experiment that better illustrates the foregoing ideas. We have a closed box, and in it there are many (say, n) different oscillators, all of different frequency. We require this to be able to distinguish our 'points,' but note that we do not know anything about their structure. We inject a particle into the box that has the following property: when it is sufficiently close to one oscillator, the oscillator increases its energy level. At the final time when we measure things, the oscillators will relax to their ground state, emitting one photon of the same frequency as the oscillator that had been excited.

The only other thing we need to assume is that somehow the signals (photons records) emitted from each oscillator can be distinguished from those emitted by the same oscillator at a different time (we need that to have spatio-temporal labels). This may be done by having, for example, a moving film around the box, and earlier or later signals from the same oscillator would be identified by different positions on the film. The fact that each oscillator may have significantly different 'half-life' before it relaxes, means that signals from different times may be confused. The important point here is that we only care about the order of photons coming from the same spatial event, since others can be distinguished by their different frequency. Finally, we will end up with a set of different records corresponding to different events, and all of them will be spatio-temporally distinguishable.

We then infuse the particle into the box, with different momentum and from different points. Each repetition (with the same initial condition) gives one possible history. We do it many times for each initial setup, and we record the results in our data-sheets (experimental protocols). After this is accomplished, we obtain the set of possible histories and their relative frequencies. This is sufficient, in our non-relativistic case, for deriving the number of different components each spatial slice has.

We may then vary slightly each oscillator separately, and repeat the experiment. By this, we will obtain information needed to recover the topology . In this setting we have the following correspondences with our theoretical scheme:

- single experiment—emitting the particle in the box with some given initial condition, and with a specific setup of the oscillators, and then record at the

[15] The order of the events is recovered, due to the fact that varying events to the past and only affects the relative frequencies of the future and NOT visa-versa.

final time the photons having been emitted (maybe read them out directly from the moving film).

- the record space—photons of different frequency on different positions on the registering film.
- a particular history—a collection of photons of different frequencies (possibly also of different positions on the film, if a moving film is required for distinguishing events in time).

To recover the configuration space, we do the following:

We assume continuity of the distribution and move slightly one oscillator at a particular 'time.' By noting the probabilities of which events are significantly altered, we define the neighborhood of this 'point.' Here we need to take into account the union of all the neighborhoods related to all the possible initial conditions. We then specify the 'temporal' and 'spatial' neighborhoods. We repeat this procedure varying slightly all the 'events' one by one. Using the proximity relation, we deduce the topology of this slice, as was described in Section 3.3.

By patching all the slices together, we **recover the topology** of our 'spacetime.'

What is Eventually Recovered. We get the effective topology of the interior of the box. This includes other objects that were not known to be there, as well as their time evolution. So if for example there was a cube of ice in the box that melted, this will be represented by a cubic hole in the configuration space that gradually changed shape to become flat.

5. CONCLUSIONS

Let us summarize what we have done. We have a laboratory in which we explore a physical system whose configuration space is unknown. We are able to run the experiments sufficiently many times, either by leaving the initial conditions unchanged, or by varying them. We also have another physical system, whose configuration space is coined RECORD SPACE. As a result of each run of the experiment, the record space acquires a state (a *quantum* state, in general). In each run, we perform a measurement over the record space. Which measurement in particular, this is a matter of our choice.

After multiple runs, we have a set of protocols (data-sheets). Each protocol tells us which events occurred within a particular experiment. This set of events is referred to as a history. When the initial conditions remain unchanged, the arising set of histories is treated as a decohering set.

Initially, as a result of our observations, we have histories and, in addition, their relative frequencies. This primary set of histories we call FIDUCIAL SET.

From the fiducial set, we deduce the number of components of our 'spacetime' (extended configuration space) as well as the number of components in each 'spatial' surface (i.e. moment of time).

We then allow for variations of the records. This yields new histories which make it possible to deduce proximity on the fiducial set and hence the topological properties of the 'spacetime.'

As a result, we reconstruct the *effective* topology of the 'spacetime' region involved in our observations. 'Effective' means that we can say nothing about the 'true' topology, and that all our statements are consequences of our observations. The working definition of configuration space that we employ is the following. CONFIGURATION SPACE is the space of all possible configurations of our system.

The topology we recover—the 'effective' one—may include holes and other topological features that result from existing 'potentials' that we do not vary. What could be referred to as the 'true' topology would be something that takes into account only the background manifold. In this sense it would be like saying that we may vary not only the initial state of the system in question, but the initial state of any potential except the gravitational (which is supposed to account for the 'geometry' of the background manifold).

We emphasize once again that we recover histories *operationalistically*. The record space is *the only* source of information we possess about the system we explore. The effective topology is then regarded as the 'best possible' (:as realistic, or as pragmatic a) picture of the actual configuration space of the system in focus as one can acquire from her 'experimental intercourse' with it.

Last but not least, some loose, anticipatory connections with the forthcoming paper (Raptis *et al.*, in press) are due here. In the latter, we develop the *relativistic* version of our 'topology-from-inverse histories' theoretical scheme. This essentially means that, due to a physical upper bound in the transmission/propagation of (material) signals, one is forced to focus more on recovering the *causal topology* of space*time* from 'inverse *causal* histories,' rather than on just recovering the topology of 'frozen, absolute, fat spatial slices' (*i.e.*, merely of '*space*') as we did presently.

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